Chapter III: Opening, Closing

Opening and Closing by adjunction

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Algebraic Opening and Closing

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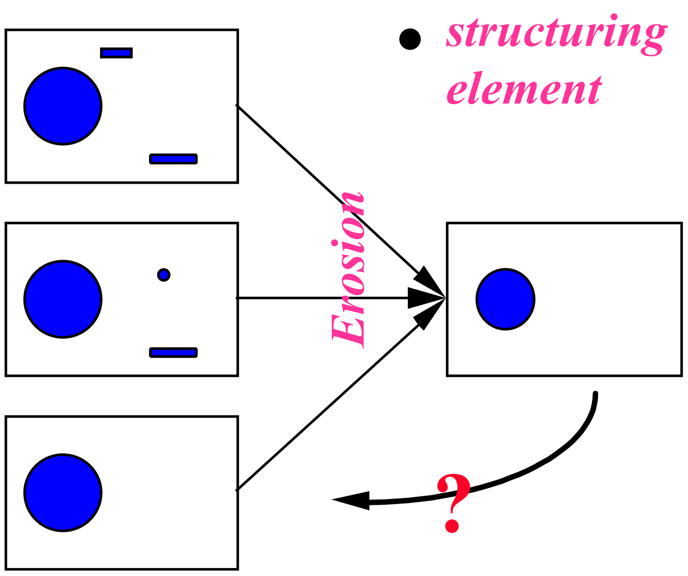
Top-Hat Transformation

Granulometry

Adjunction Opening and Closing

The problem of an inverse operator

Several different sets may admit a same erosion, or a same dilate. But among all possible inverses, there exists always a smaller one (a larger one). It is obtained by composing erosion with the adjoint dilation (or *vice versa*) .



The mapping is called ***adjunction opening***, and is denoted by

$\gamma_B = \delta_B \varepsilon_B$ *(general case)*

$X \circ B = [(X \ominus B) \oplus B]$ (τ-*operators)*

By commuting the factors δ_B and ε_B we obtain the ***adjunction closing***

$\varphi_B = \varepsilon_B \delta_B$ *(general case)*,

$X \bullet B = [X \oplus B) \ominus B]$ (τ-*operators)*.

*(These operators, due to G. Matheron are sometimes called **morphological**)*

Properties of Adjunction Opening and Closing

Increasingness

Adjunction opening and closing are increasing as products of increasing operations.

(Anti-)extensivity

By doing $Y = \delta_B(X)$, and then $X = \varepsilon_B(Y)$ in adjunction $\delta_B(X) \subseteq Y \Leftrightarrow X \subseteq \varepsilon_B(Y)$, we see that:

$\delta_B \varepsilon_B (X) \subseteq X \subseteq \varepsilon_B \delta_B (X)$ hence $\varepsilon_B (\delta_B \varepsilon_B) \subseteq \varepsilon_B \subseteq (\varepsilon_B \delta_B)\varepsilon_B \Rightarrow$ **$\varepsilon_B \delta_B \varepsilon_B = \varepsilon_B$**

Idempotence

The erosion of the opening equals the erosion of the set itself. This results in the idempotence of γ_B and of φ_B :

$\varepsilon_B (\delta_B \varepsilon_B) = \varepsilon_B \Rightarrow \delta_B \varepsilon_B (\delta_B \varepsilon_B) = \delta_B \varepsilon_B$ *i.e.*

$\gamma_B \gamma_B = \gamma_B$ and,
by duality $\varphi_B \varphi_B = \varphi_B$

Finally, if $\varepsilon_B(Y) = \varepsilon_B(X)$, then $\gamma_B(X) = \delta_B \varepsilon_B(X) = \delta_B \varepsilon_B(Y) \subseteq Y$. Hence, γ_B is the smallest inverse of erosion ε_B .

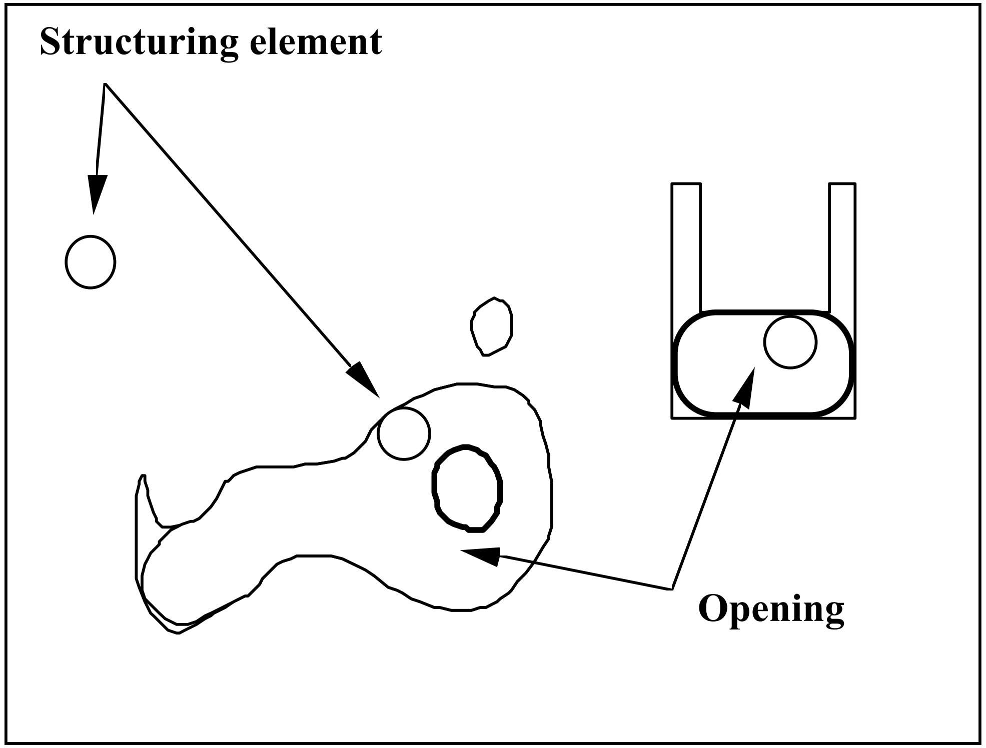
Amending Effects of Adjunction Opening

Geometrical interpretation

$z \in \gamma_B(X) \Leftrightarrow \quad z \in B_y$ and $\quad y \in X \ominus B$

hence **$z \in \gamma(X) \Leftrightarrow z \in B_y \subseteq X$**

- the opened set $\gamma_B(X)$ is the union of the structuring elements B(x) which are included in set X.
- In case of a τ–opening, $\gamma_B(X)$ is the zone swept by the structuring element when it is constrained to be included in the set.



When B is a disc, the opening amends the caps, removes the small islands and opens isthmuses.

Amending Effects of Adjunction Closing

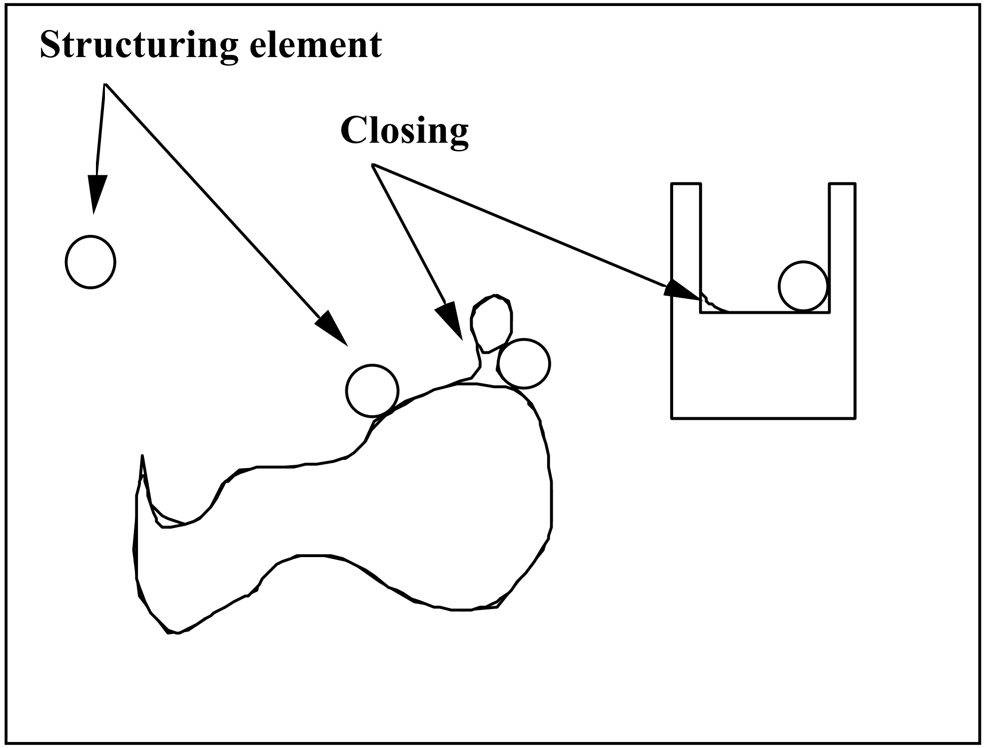
Geometrical interpretation

- By taking the complement in the definition of XoB we see that

$$X \bullet B = [(X \oplus \check{B}) \ominus \check{B}]$$

- The τ-closing is the complement of the domain swept by B as it misses set X. Note that in most of the practical cases, set B is symmetrical, *i.e.* identical to $\check{B}$.

- Note that when a shift affects both erosions and dilations, it does not acts on openings and closings.



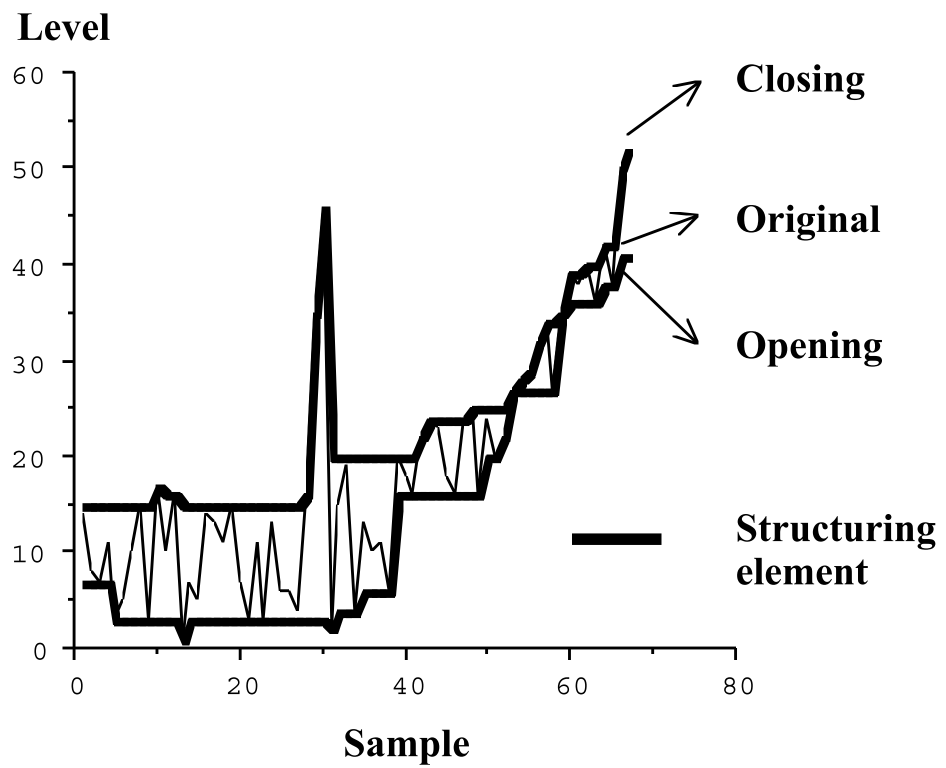
When B is a disc, the closing closes the channels, fills completely the small lakes, and partly the gulfs.

Effects on Functions

- The adjunction opening and closing create a ***simpler function*** than the original. They smooth in a nonlinear way.

- The ***opening*** (closing) removes ***positive*** (negative) ***peaks*** that are thinner than the structuring element.

- The opening (closing) remains below (above) the original function.



Algebraic Opening and Closing

The three basic properties of adjunction openings $\delta\varepsilon$ and closings $\varepsilon\delta$ are also the **axioms** for the algebraic notion of an opening and a closing.

Definition **:** In algebra, any transformation which is:

- increasing, anti-extensive and idempotent is called an (algebraic) opening,
- increasing, extensive and idempotent is called a (algebraic) closing.

Particular cases **:**

Here are two very easy ways for creating algebraic openings and closing, they are often used in practice:

1) Compute various opening (closing) and take their supremum (or the infimum in case of closings).

2) Use a ***reconstruction*** process (see V- 9 and VI - 5)

Invariant Elements

Let $\mathcal{B}$ be the image of lattice L under the algebraic opening γ, *i.e.* $\mathcal{B} = \gamma(L)$. Since γ is idempotent, set $\mathcal{B}$ generates the family of ***invariant sets*** of γ :

$$\mathbf{b} \in \mathcal{B} \iff \gamma(\mathbf{b}) = \mathcal{B} .$$

1/ Classe B is ***closed under sup***. For any family $\{b_j, j \in J\} \subseteq \mathcal{B}$, we have

$$\gamma(\vee b_j, j \in J) \geq \vee\{\gamma(b_j), j \in J\} = \vee(b_j, j \in J)$$

by increasingness, and the inverse inequality by anti-extensivity de γ. Moreover , $0 \in \mathcal{B}$. Note that γ does not commute under supremum.

2/ Therefore, γ is the ***smallest extension*** to L of the identity on $\mathcal{B}$, *i.e.*

$$\gamma(\mathbf{x}) = \vee\{\mathbf{b} : \mathbf{b} \in \mathcal{B}, \mathbf{b} \leq \mathbf{x}\}, \qquad \mathbf{x} \in \mathbf{L} .$$

[The right member is an invariant set of γ smaller than x, but also that contains γ (x).]

Representation of openings

Conversely, let $\mathcal{B}$ be the class closed under sup. that an *arbitrary* class $\mathcal{B}_0$ generates. With each $b \in \mathcal{B} \cup 0$ associate dilation

$$\delta_b(a) = b \quad \text{when} \quad a \geq b \quad ; \quad \delta_b(a) = 0 \quad \text{when not} .$$

By adjunction, it yields opening

$$\gamma_b(x) = \vee \ \{\delta_B(a) \ , \ \delta_b(a) \leq x \ \} = \begin{cases} b & \text{when } b \leq x , \\ 0 & \text{when not} . \end{cases} \qquad (2)$$

[If $b \leq x$, it suffices to take $a \geq b$ for obtaining a term $= b$ in the $\vee$; if not, no $\delta_b(a) \neq 0$ is $\leq x$] . Then relation (1) becomes

$$\boldsymbol{\gamma(x) = \vee\{ \delta_b(a) \ , \ b \in \mathcal{B} \}.}$$

Theorem (G. Matheron , J. Serra) : 1/ With any class $\mathcal{B}_0 \subseteq L$ corresponds a unique opening on L that has for invariant sets the $b \in \mathcal{B}_0$.

2/ Every algebraic opening γ on lattice L may be represented as the union of the adjunction openings linked to the invariant sets of γ by relation (2).

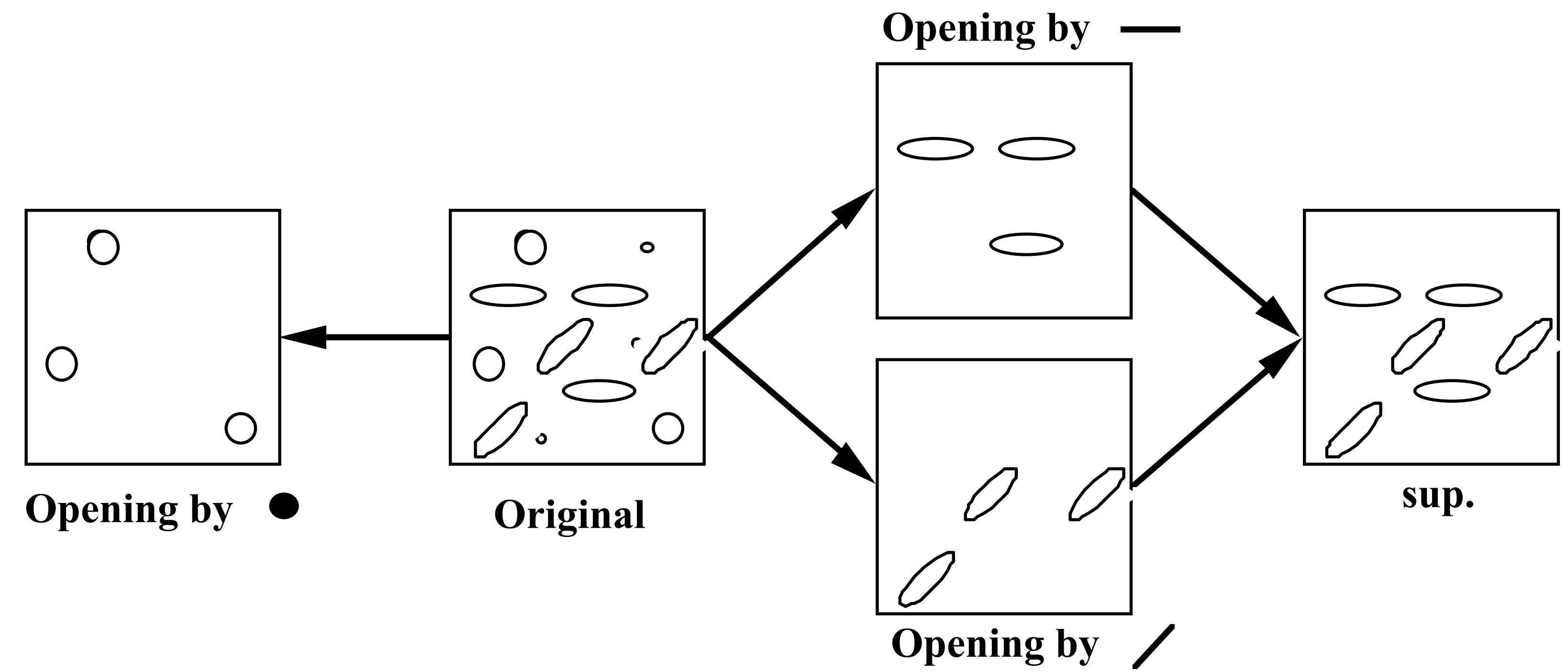
Suprema of Openings

Theorem **:**

- Any supremum of openings is still an opening.
- Any infimum of closings is still a closing.

Application **:**

For creating openings with specific selection properties, one can use structuring elements with various shapes and take their supremum.



"Top-hat" Transformation

Goal

- The "top-hat" transformation, due to F. Meyer, aims to suppress slow trends, therefore to enhance the contrasts of some features in images, according to size or shape criteria. This operator serves mainly for numerical functions.

Definition

- The "top-hat" transformation is the residue between the **identity** and a (compatible with vertical translation) **opening**

$$\rho(f) = f - \gamma(f) \quad (\text{ functions}) \quad ; \quad \rho(X) = X \setminus \gamma(X) \quad (\text{sets})$$

- A dual top-hat can be defined: the residue between a **closing** and the **identity:**

$$\rho^*(f) = \varphi(f) - f \quad (\text{ functions}) \quad ; \quad \rho^*(X) = \varphi(X) \setminus X \ (\text{sets})$$

Top-hat Properties

Idempotence

- The top-hat is idempotent (but not increasing). Moreover, when the original signal is positive the top hat is anti-extensive:

$$\rho(\rho(f)) = \rho(f) \qquad\qquad f > 0 \;\Rightarrow \rho(f) < f$$

Geometrically speaking, the top-hat reduces to zero the ***slow trends*** of the signal.

Robustness

- If Z stands for the set of points where the opening is strictly smaller than f, *i.e.*

$$Z = \{\, x : (\gamma f)(x) < f(x) \,\}$$

and if g is a positive function whose support is included in Z, then

$$T(g) = g \qquad \text{and} \qquad T(f+g) = T(f) + T(g)$$

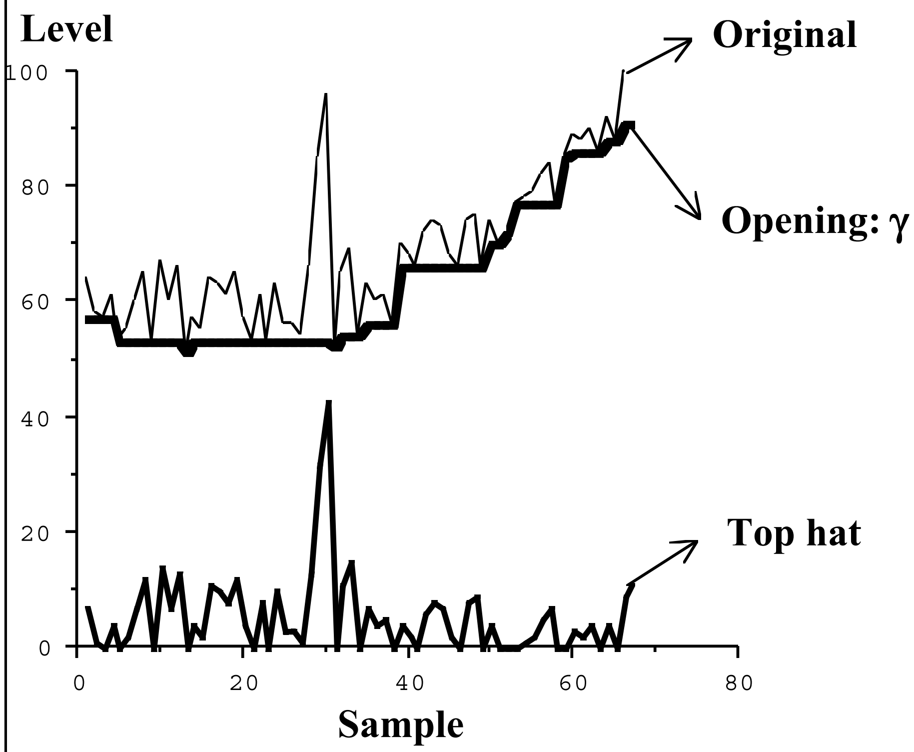
Use of the Top-hat

Sets

- The top-hat extracts the objects that have not been eliminated by the opening. That is, it removes objects larger than the structuring element.

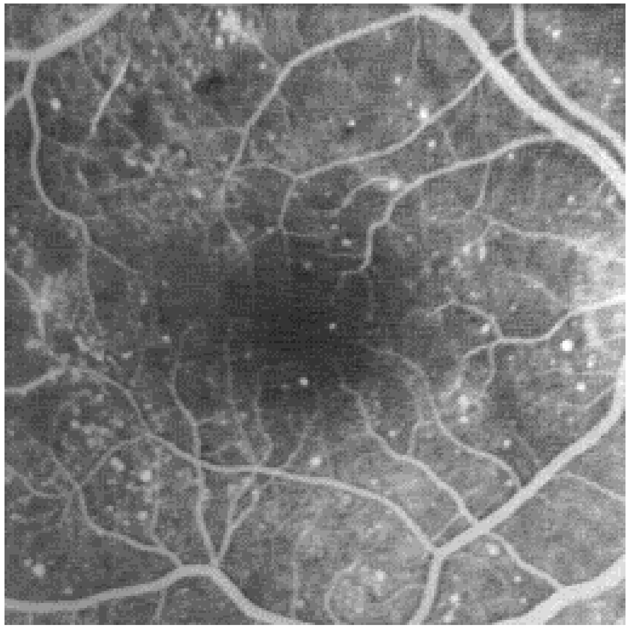
Functions

- The top-hat is used to extract contrasted components with respect to the background. The basic top-hat extracts positive components and the dual top hat the negative ones.
- Typically, top-hats remove the slow trends, and thus performs a contrast enhancement.

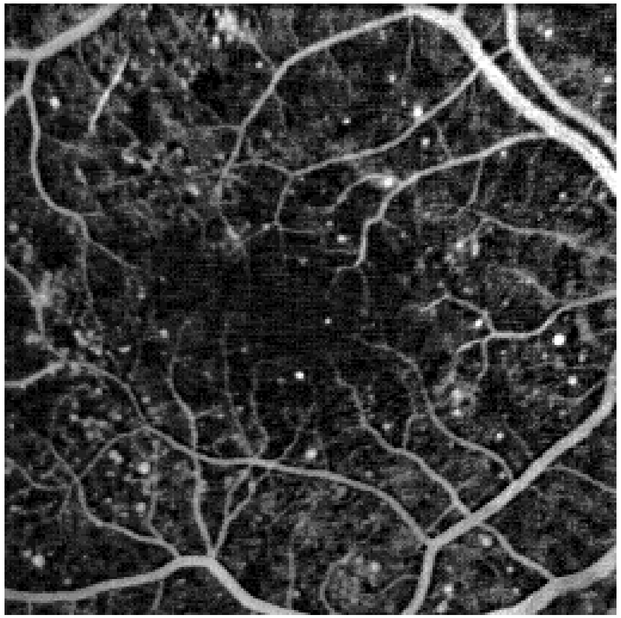


An Example of Top-hat

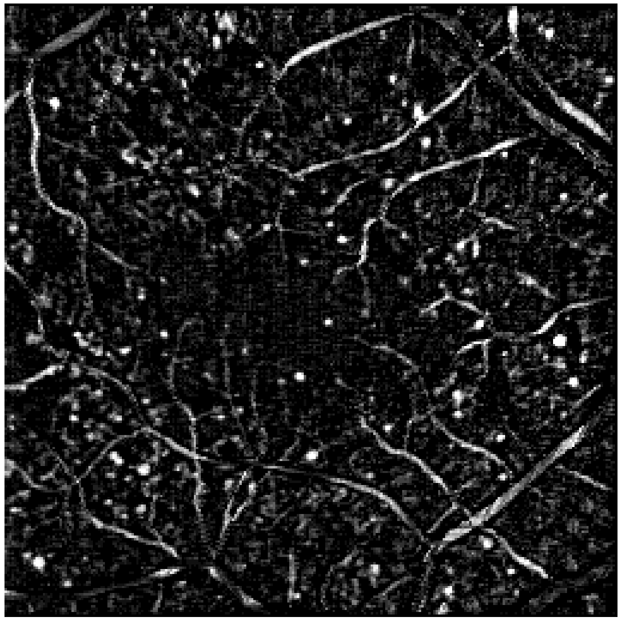
Comment : *The goal is the extraction of the aneurisms (the small white spots). Top hat c), better than b) is far from being perfect. Here opening by reconstruction yields a correct solution (VI-7).*



Negative image of the retina.



Top hat by an hexagon opening of size 10.



Top hat by the sup of three segments openings of size 10.

Granulometry: an Intuitive Approach

- ***Granulometry*** is the study of the size characteristics of sets and of functions. In physics, granulometries are generally based on **sieves** ψ_λ of increasing meshes $\lambda > 0$. Now,
  - by applying sieve λ to set X, we obtain the over-sieve $\psi_\lambda(X) \subseteq X$;
  - if Y is another set containing X, the Y-over-sieve, for every λ, is larger than the X-over-sieve, *i.e.* $X \subseteq Y \Rightarrow \psi_\lambda(X) \subseteq \psi_\lambda(Y)$;
  - if we compare two different meshes λ and μ such that $\lambda \geq \mu$, the μ-over-sieve is larger than the λ–over-sieve, *i.e.* $\lambda \geq \mu \Rightarrow \psi_\lambda(X) \subseteq \psi_\mu(X)$
  - finally, by applying the largest mesh λ to the μ–over-sieve, we obtain again the λ–over-sieve itself, *i.e.* $\psi_\lambda \psi_\mu (X) = \psi_\mu \psi_\lambda (X) = \psi_\lambda (X)$
- Such a description of the physical sieving suggests to resort to **openings** for an adequate formalism. The sizes of the structuring elements will play the role of the the sieves meshes.

Granulometry: a Formal Approach

- ***Matheron Axiomatics*** defines a granulometry as a family $\{\gamma_\lambda\}$

  i) of openings depending on a positive parameter λ,

  ii) and which decrease as λ increases: $\lambda \geq \mu > 0 \Rightarrow \gamma_\lambda \leq \gamma_\mu$.

This second axiom is equivalent to the **semi-group** where the composition of two operations is equal to the stronger one, namely

$$\gamma_\lambda \gamma_\mu = \gamma_\mu \gamma_\lambda = \gamma_{\sup(\lambda,\mu)} \qquad \mathbf{(1)}$$

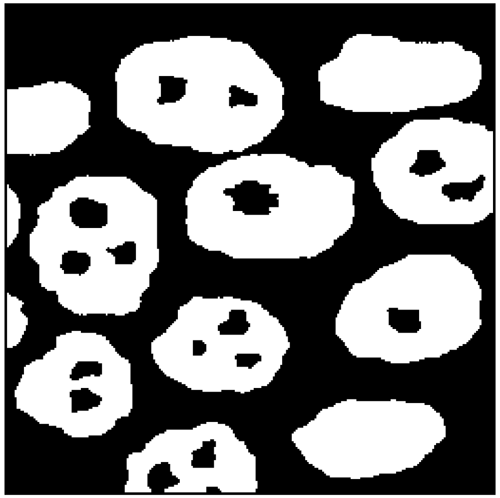
[{ $\lambda \geq \mu > 0 \Rightarrow \gamma_\lambda \leq \gamma_\mu$ } $\Rightarrow$ $\gamma_\lambda = \gamma_\lambda \gamma_\lambda \leq (\gamma_\lambda \gamma_\mu \vee \gamma_\mu \gamma_\lambda) \leq \gamma_\lambda$;
conversely, $\gamma_\lambda = \gamma_\mu \gamma_\lambda$ et $\gamma_\lambda \leq I \Rightarrow \gamma_\lambda \leq \gamma_\mu$ hence semi-group (1).]

- If $\mathcal{B}_\lambda$ et $\mathcal{B}_\mu$ stand for the invariant elements of γ_λ and of γ_μ respectively, then we easily see that

$$(1) \Leftrightarrow \mathcal{B}_\lambda \subseteq \mathcal{B}_\mu$$

- Finally, when γ_λ's are **adjunction openings**, *i.e.* $\gamma_\lambda(X)=X\circ\lambda B$ (with ***similar*** structuring elements), then the granulometry axioms are fulfilled if and only if the structuring element B is compact and ***convex***.

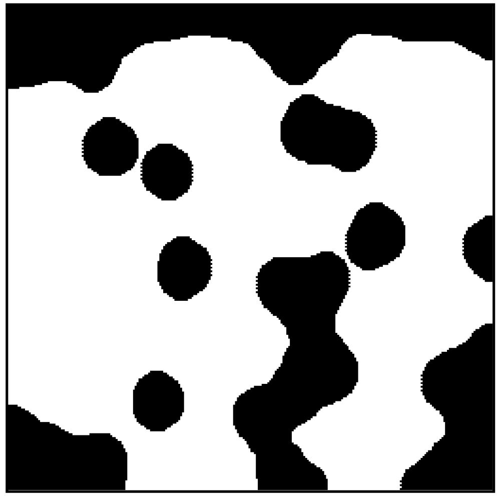
An Example of Granulometry

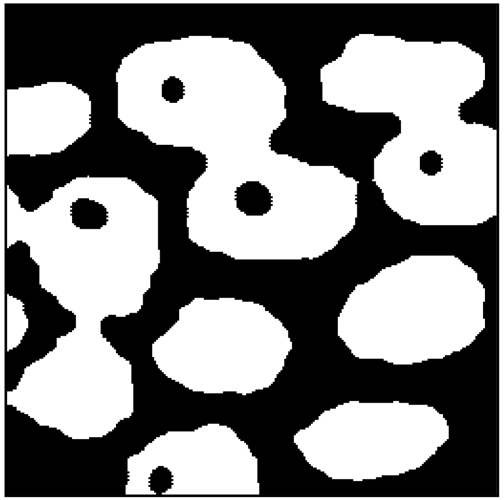


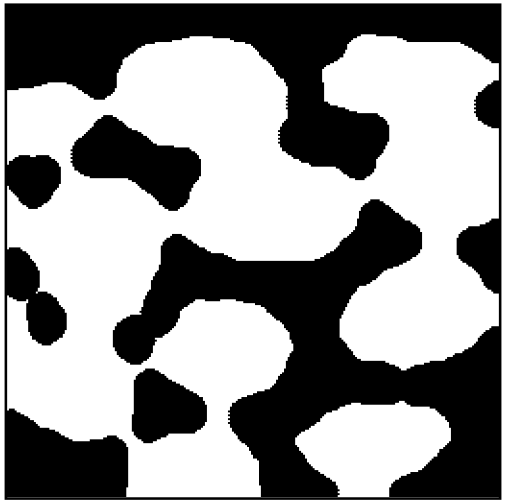
By duality, the families of closings $\{\varphi_\lambda , \lambda>0\}$ increasing in λ generate ***anti-granulometries*** of law

$$\varphi_\lambda \varphi_\mu = \varphi_\mu \varphi_\lambda = \varphi_{\sup(\lambda,\mu)} .$$

Here, from left to right, closings by increasing discs.





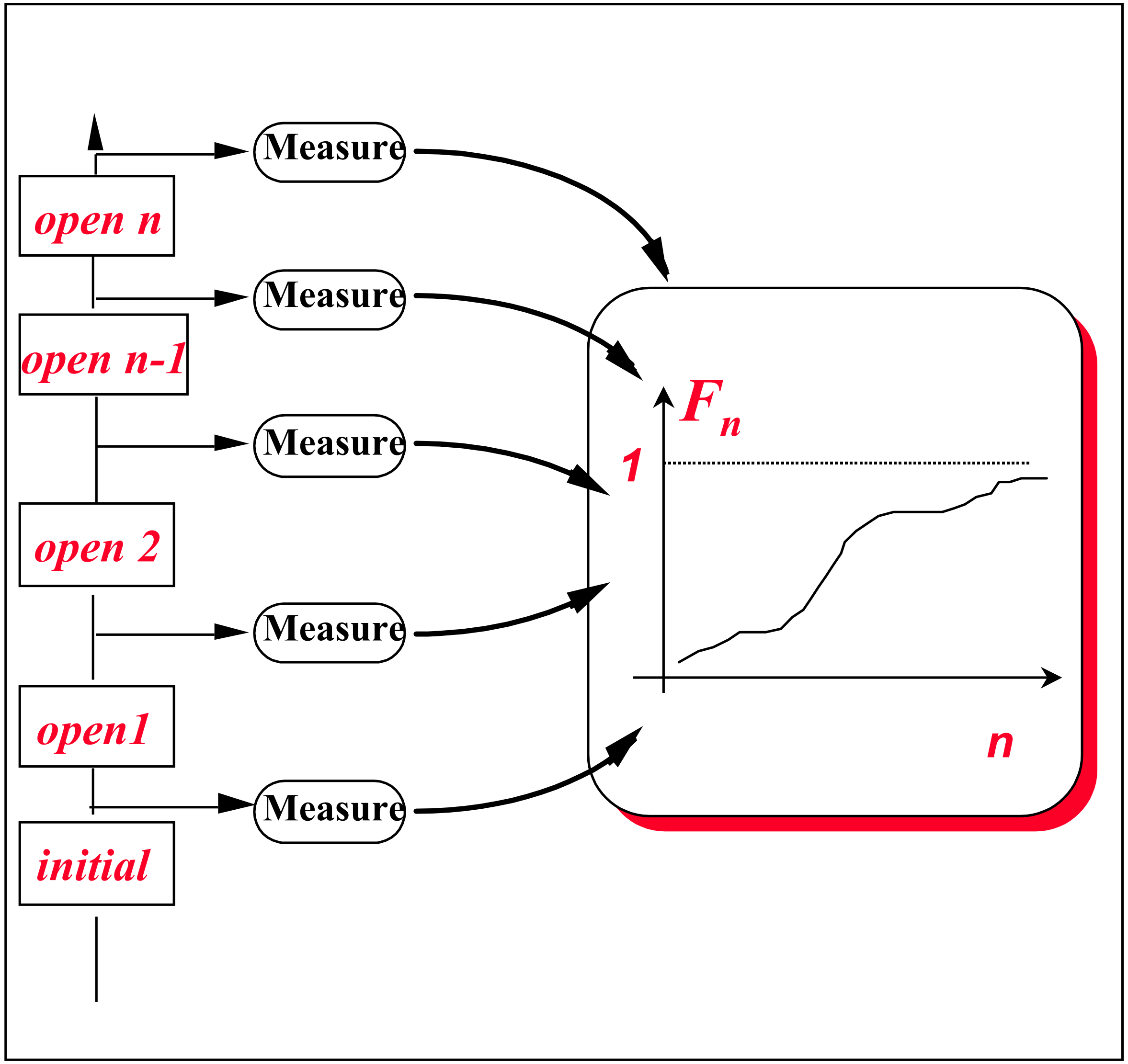


Granulometry and Measurements

- A granulometry is computed from a pyramid of openings, or closings, whose each element is given a size, λ say;
- Value λ is the similarity ratio holding on the involved structuring element(s) .
- At the output of each filter, the area is measured (set case), or the integral in case of functions, M_λ say. Then, the monotonic curve

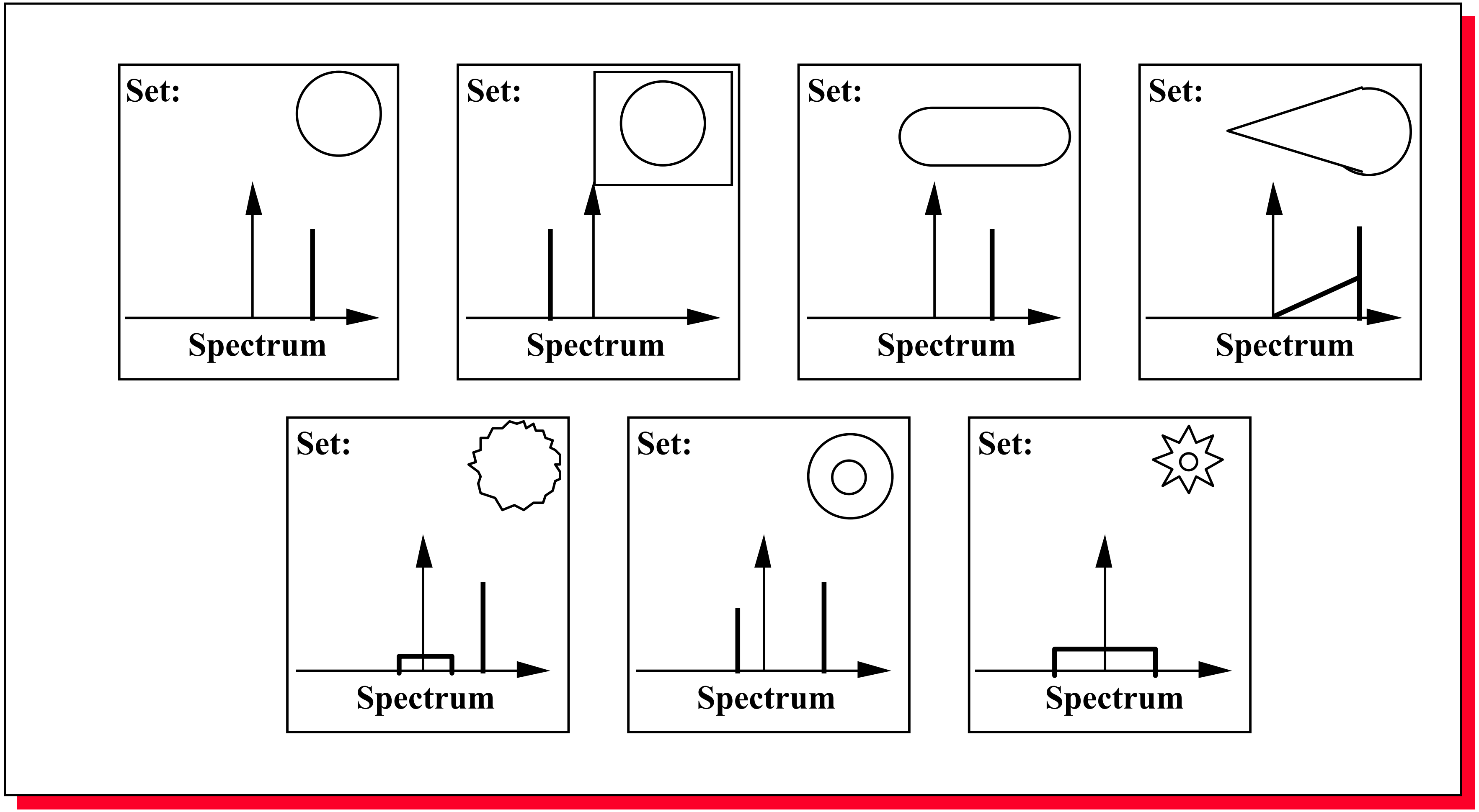
$$F_\lambda = 1 - M_\lambda / M_0$$

is a ***Distribution Function*** **.**



Granulometric Spectrum

One also uses the *granulometric spectrum*, that is the derivative of the granulometric distribution function.



References

On Adjunction Opening, and on Granulometry :

- The notion of an adjunction opening (or closing) and that of a granulometry were introduced by G.Matheron in 1967 {MAT67}, for the binary and translation invariant case. In 1975, {MAT75} linked them with algebraic closings in E.H.Moore's sense (begining of the XXth century) and generalized the notion of a granulometry, but still in the set case.The extension to lattices, with the representation theorem, is due to J. Serra {SER88. For details and examples about granulometric spectrum, see {HUN75},{SER82,ch.10},{MAR87a}.For algebraic openings, see{RON93}.

On Gradient and Top hat mappings :

- Morphological gradient appears for the first time with S.Beucher and F.Meyer in 1977 {BEU77} and top hat operator with F.Meyer in 1977 {MEY77}, followed by a reformulation in terms of function mappings in {SER82,ch.12}. More sophisticated versions of the gradient can be found in {BEU90}, and top hat mappings with non flat structuring elements are proposed in {STE83}, (see also {SER88, ch9}).