Chapter II : Dilation, Erosion

Set Erosion and Dilation

1 - General Case
2 - Minkowski Operations
3 - Standard Operators

Erosion and Dilation for Functions

Planar Operations

Gradient and Laplacian

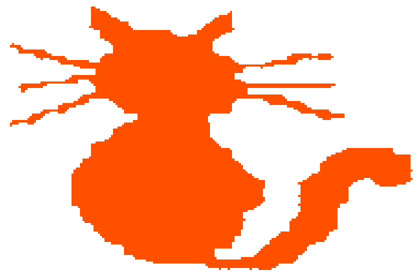
H.M.T. and Set Erosion

- The objects under study are here the sets $X \subseteq E$. The theory of mathematical morphology describes them
  - by associating with all $x \in E$ two probes $A(x)$ and $B(x)$;
  - and by testing whether one is contained in X and the other in X^c.
- The functions $x \rightarrow A(x)$ and $x \rightarrow B(x)$ are called **structuring elements**, and the Hit-or-Miss transform of X (**H.M.T.**) is defined by the operation (Ch. 8)

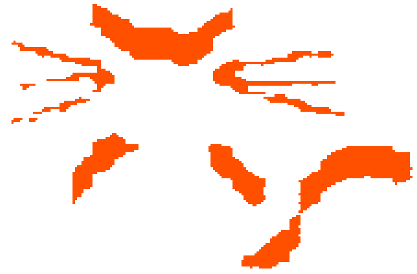
$$\eta(X) = \{z : A(z) \subseteq X^c ;\ B(z) \subseteq X\}$$

- When $A = \emptyset$, $\eta(X)$ becomes **the eroded** of X by the (variable) structuring element B. One writes

$$\varepsilon_B(X) = \{z : B(z) \subseteq X\}$$



Initial image



example of H.M.T.

Adjunction (I)

- ***Set Erosion :*** Operation ε_B commutes under $\cap$:

$$\boldsymbol{\varepsilon_B(\cap X_i)} = \{z: B(z) \subseteq \cap X_i\} = \cap \{z: B(z) \subseteq X_i\} = \boldsymbol{\cap\, \varepsilon_B(X_i)},$$

Therefore, it is effectively an **erosion.**

- ***Adjunction :*** The equivalences

$$X \subseteq \varepsilon_B(Y) \quad \Leftrightarrow \quad \{x \in X \Rightarrow B(x) \subseteq Y\} \quad \Leftrightarrow \quad \cup\{B(x), x \in X\} \subseteq Y$$

yield the operation

$$\boldsymbol{\delta_B(X) = \cup \{B(x), x \in X\}}$$

which commutes under $\cup$. The later is thus a **dilation**, said to be **adjoint** of ε. Adjunction is an involution, since by taking the inverse way, we see that ε adjoint of δ.

- ***Structuring Element :*** Since $\delta_B(X) = \cup\{\delta_B(x), x \in X\}$, the mapping ***" structuring element "*** $\boldsymbol{x \rightarrow \delta_B(x) = B(x)}$ suffices to characterise both

  - dilation $\delta : X \rightarrow \delta(X)$ - and erosion $\varepsilon : X \rightarrow \varepsilon(X)$.

Adjunction (II)

The Adjunction Theorem *(E. Gallois….H .Heijmans, Ch. Ronse, J. Serra)***:**
When two operators δ and ε are linked by the equivalence

$$\mathbf{X \subseteq \varepsilon (Y) \quad \Leftrightarrow \quad \delta (X) \subseteq Y}$$

then they necessarily form an "***erosion-dilation***" doublet.

- *Proof* : Let be a family Y_i , $i \in I$, and X such that

$$\delta(X) \subseteq \cap Y_i \qquad \Leftrightarrow \qquad \delta(X) \subseteq Y_i \quad \text{for every } i \in I ,$$

By adjunction : first inclusion $\Leftrightarrow \quad X \subseteq \varepsilon (\cap Y_i)$

second inclusion $\Leftrightarrow \quad X \subseteq \varepsilon_B (Y_i)$, $i \in I$, $\Leftrightarrow \quad X \subseteq \cap\varepsilon (Y_i)$

This implies $\varepsilon (\cap Y_i) = \cap\varepsilon (Y_i)$, *i.e.* that ε is an erosion (*id.* for the dilation).

First Representation *(J. Serra)* **:** For any pair (δ,ε) we have :

$$\varepsilon (Y) = \cup\{ X : \delta (X) \subseteq Y\} \qquad \delta (X) = \cap\{ Y : \varepsilon (Y) \subseteq X\}$$

Curiously, erosion appears here as a union and dilation as an intersection

N.B. *the approach extends to mappings from one lattice into another.*

Representations and Semi-groups

- ***Second Représentation. Theorem*** *(J.Serra)* **:** Every **increasing** mapping ψ on $\mathcal{P}(E)$ can be written as a union of erosions as follows

$$\psi = \cup \{ \varepsilon_B , B \in \mathcal{P}(E) \} ,$$

with $\varepsilon_B(X) = \psi(B)$ if $X \supseteq B$, and $\varepsilon_B(X) = \emptyset$ otherwise (dual result for the dilation).

This representation generalises G. Matheron's one, for the translation invariant case (II, 14), and extends itself to the complete lattice case.

- ***Semi-groups*:** The composition product of two dilations (resp. erosions) is still a dilation (resp. erosion). Indeed

$$\delta_{B2}\delta_{B1}(X) = \cup \{ B_2(y) , y \in \cup \{ B_1(x) , x \in X \} = \cup \{ \delta_{B2}[B_1(x)] , x \in X \}$$

hence $\delta_{B2}\delta_{B1} = \delta_A$; $\varepsilon_{B2}\varepsilon_{B1} = \varepsilon_A$ **with** $A = \delta_{B2}(B_1)$

[*Semi-group* $\Rightarrow$ *no inverse* $\Leftrightarrow$ ***loss*** *of information*.]

Case of Set Translation Invariance

- Suppose set E equipped with a translation τ. The translation invariant operations ψ: $\mathcal{P}(E) \rightarrow \mathcal{P}(E)$ are called **τ-mappings** .

- Then, the two basic dilations on $\mathcal{P}(E)$ are
  - the **Minkowski Addition** , which is the **unique** τ-dilation,
  - the **Geodesic Dilation**, which is limited to a given mask .

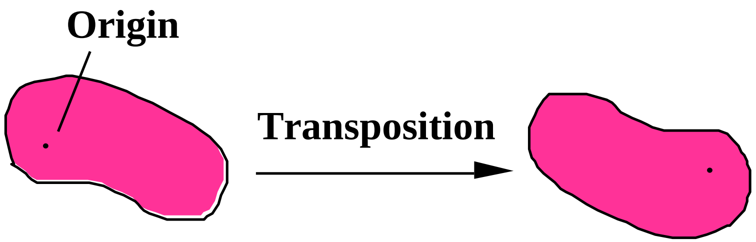
- for all $X \subseteq E$, introduce:

1) set X_b , **translate** of X according to vector b :

$$X_b = \{x+b, x \in X\}$$

2) set $\check{X}$, **transposed** or **reflected** of X :

$$\check{X} = \{ -x , x \in X\}$$



we have: $x \in \check{B}_z \Leftrightarrow z - x \in B$. Note that B is **symmetrical** when it is equal to its transposed.

Set Dilation and Minkowski Addition

• The τ-dilations are called Minkowski Additions. Each of them is characterized by the transform B of the origin, which turns out to be the basic Structuring Element. By putting $\delta_B(X) = X \oplus B$, we have

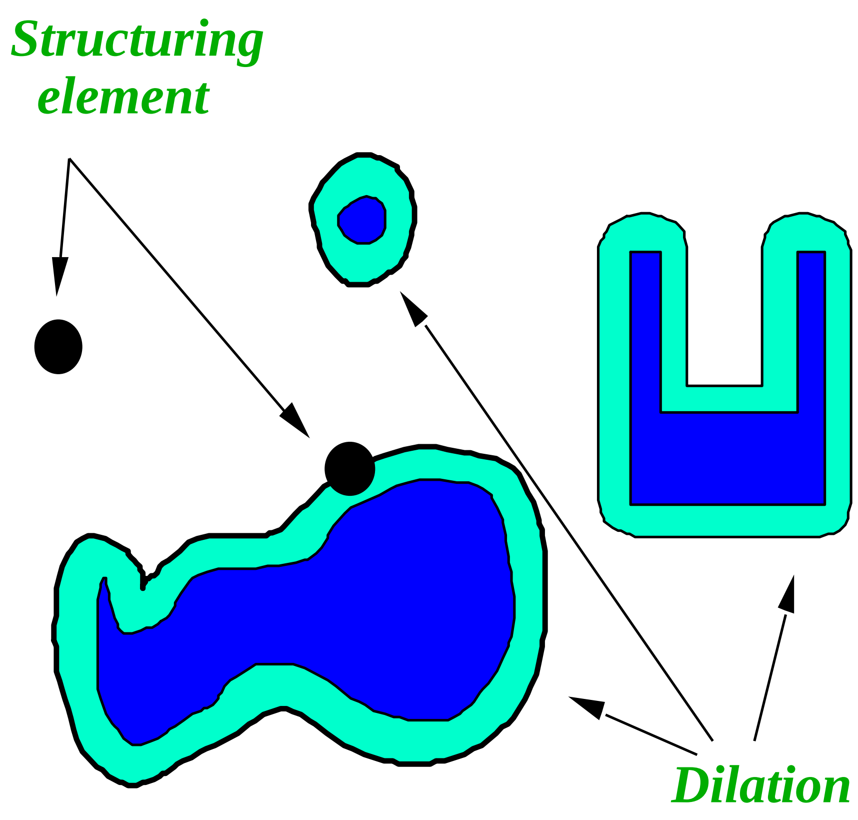
$$X \oplus B = \cup \{ B_x ,\ x \in X\} = \cup \{ x + b,\ x \in X,\ b \in B \} = \cup \{ X_b ,\ b \in B \} = B \oplus X$$

from $z \in \delta_B(X) \Leftrightarrow \{ b = z - x \in B \text{ et } x \in X \}$

$\Leftrightarrow \{ \exists x : x \in \check{B}_z \cap X \}$

we draw that the dilate of X by B is the locus of those points z such that the transposed set $\check{B}_z$ hits X :

$$\delta_B(X) = \{ z : \check{B}_z \cap X \neq \emptyset \}$$



Set Erosion and Minkowski Subtraction

- The Minkowski subtraction of X by B is, by definition, the erosion X⊖B *adjoint* to X⊕B .
- ***Geometrical interpretation***

X⊖B turns out to be the locus of the positions of the centre z of the structuring element B_z when the latter is included in X :

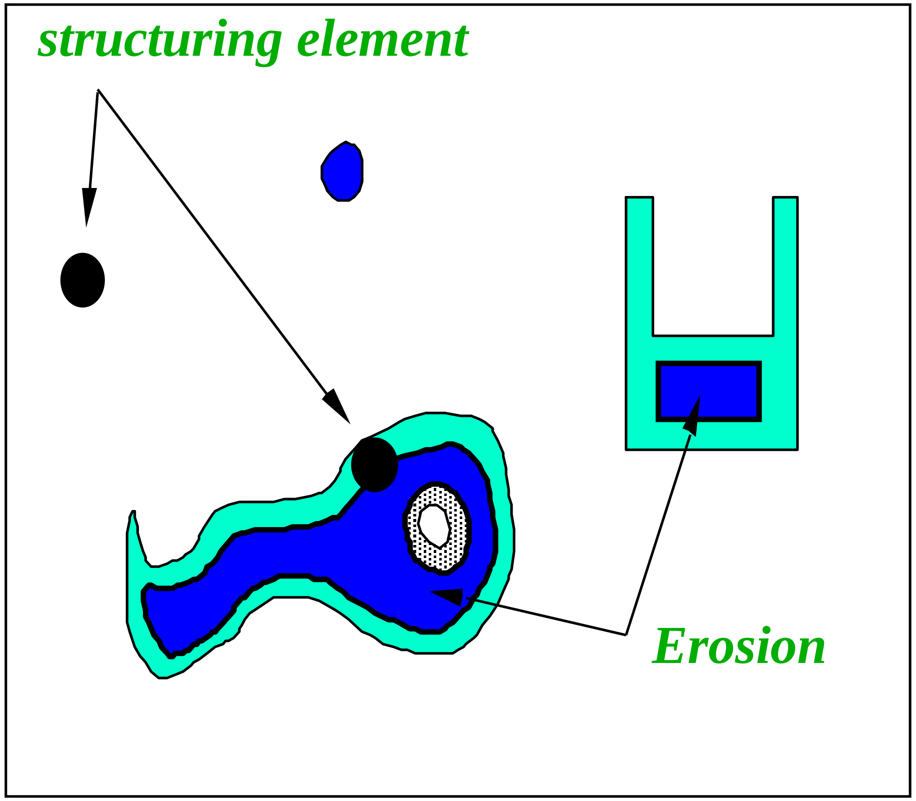
$$\varepsilon_B(X) = X \ominus B = \{ z : B_z \subset X \}$$

- ***∩ - Representation***

$B_z \subseteq X \Leftrightarrow \forall b \in B:\ b+z \in X$

$\Leftrightarrow \forall b \in B:\ z \in X_{-b}$, hence

$$X \ominus B = \cap \{ X_b, b \in \check{B} \}$$



The two Dualities

- *Adjunction,* already seen, is the duality

$$\mathbf{X \subseteq Y \ominus B \quad \Leftrightarrow \quad X \oplus B \subseteq Y \qquad X, Y \in E .}$$

It characterises the pairs "erosion-dilation". The adjoint term looks like an inverse. In particular, when X, Y et B are **convex and similar**, then

$$X = Y \ominus B \Leftrightarrow X \oplus B = Y.$$

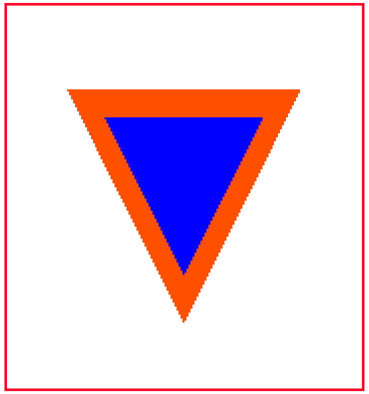
- Another duality is obtained by taking the ***complement*** *i.e.* in case of an erosion, by putting :

$$\psi(X) = (X^c \ominus B)^c \quad .$$

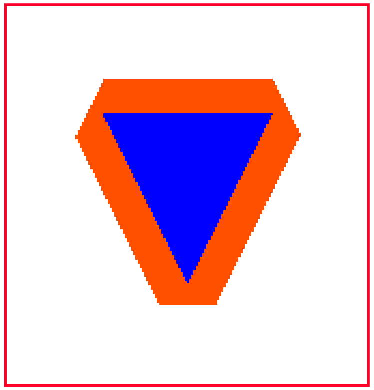
Now, $(X^c \ominus B)^c = [\cap \{(X_b)^c, b \in \check{B}\}]^c = \cup \{X_b, b \in \check{B}\}$

i.e. $\mathbf{\psi(X) = (X^c \ominus B)^c = X \oplus \check{B}}$.

The operation dual, under complement, of Minkowski subtraction by B is Minkowski addition by $\check{B}$.



Dilate of X par B, similar to each other



Dilate of the same X by the transposed $\check{B}$

Algebraic Properties of Minkowski Operations

Distributivity

We have the following ***equalities***

$X \oplus (B \cup B') = (X \oplus B) \cup (X \oplus B')$

$X \ominus (B \cup B') = (X \ominus B) \cap (X \ominus B')$

$(X \cap Z) \ominus B = (X \ominus B) \cap (Z \ominus B)$

but only the ***inclusions***

$X \oplus (B \cap B') \subseteq (X \oplus B) \cap (X \oplus B')$

$X \ominus (B \cap B') \supseteq (X \ominus B) \cup (X \ominus B')$

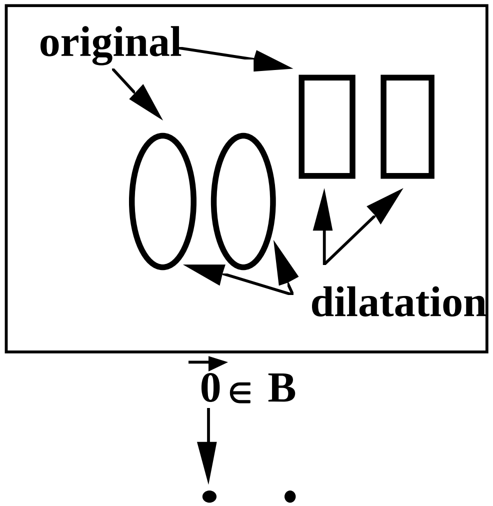
$(X \cup Z) \ominus B \supseteq (X \ominus B) \cap (Z \ominus B)$

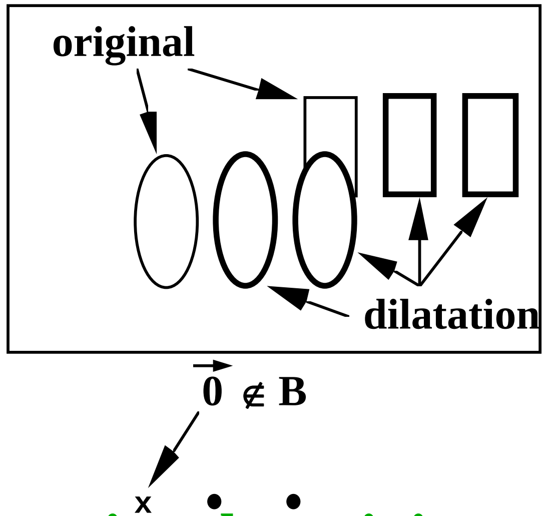
Extensivity

$O \in B \Rightarrow$

$\mathbf{X \subseteq (X \oplus B)}$

$\mathbf{(X \ominus B) \subseteq X}$





Dilation is extensive and erosion anti-extensive if **B contains the origin**

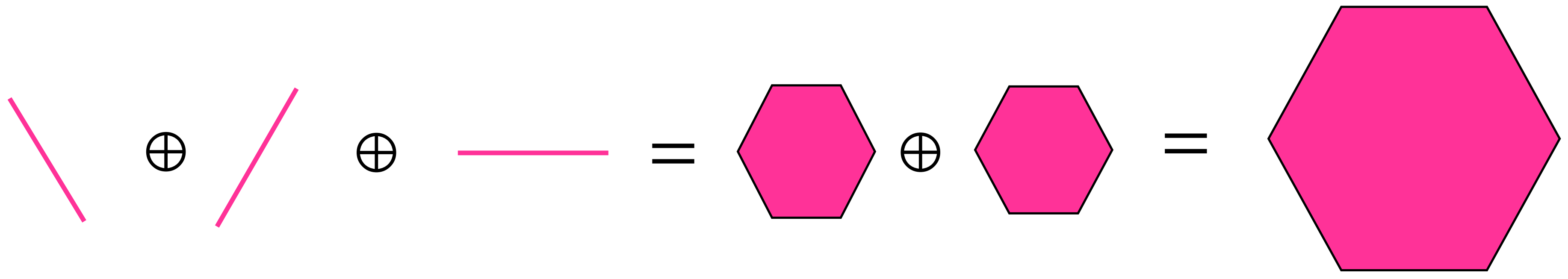
Minkowski Addition by Convex sets

- In the Euclidean space R^n denote by λB the set similar of B by factor λ. Then the semi-goup law:

$$[(X \oplus \lambda B) \oplus \mu B)] = X \oplus (\lambda + \mu) B$$

is satisfied if and only if B is **compact convex** ($x,y \in B \Rightarrow [x,y] \in B$). Moreover, if B is plane and symmetrical, it is equal to a product of dilations by **segments**.

- Practically, the dilation (*resp*. the erosion) of a set X by the convex structuring element λB reduces to λ dilations (*resp*. erosions) by the structuring element B. Iteration acts as a magnification factor.



Edge Effects

Most of the scenes under study are restrictions, to a rectangle Z, of a larger set X.

- Experimentally, one can access only $X \cap Z$, or $X \cup Z^c$, according to the value 0 ou 1 that one decide to give to the outside. Now, for B symmetrical

$$\mathbf{(X \cap Z) \ominus B = (X \ominus B) \cap (Z \ominus B)} \quad \text{and}$$

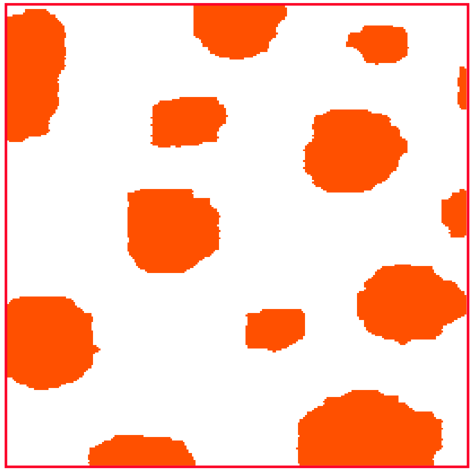
$$(X \cup Z^c) \oplus B = (X \oplus B) \cup (Z^c \oplus B) = (X \oplus B) \cup (Z \ominus B)^c$$

hence

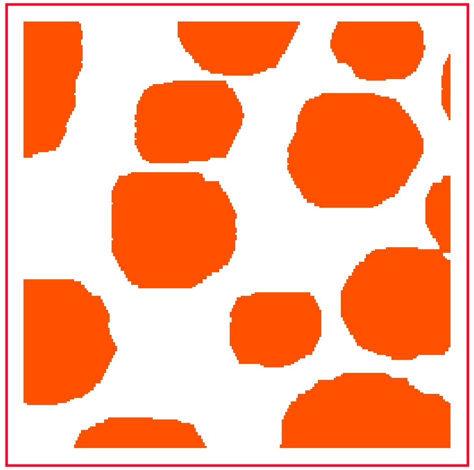
$$\mathbf{[(X \cup Z^c) \oplus B] \cap (Z \ominus B) = (X \oplus B) \cap (Z \ominus B)}$$

- In other words, the transforms $(X \oplus B)$ et $(X \ominus B)$ are correctly known inside ***mask Z eroded*** itself by B.

Worse, when we concatenate a sequence of transformations we soon reduce the mask to ∅ !



Initial set $\mathbf{(X \cap Z)}$



Dilate $\mathbf{(X \oplus B) \cap (Z \ominus B)}$

Standard Dilation et Erosion

- To solve the problem, we will reduce progressively the structuring element when it comes near the edge. We (progressively...) loose translation invariance, but the result is provided in the ***whole mask*** Z .

- In such a "**standard**" approach, where the structuring element $x \rightarrow B_x$ becomes $x \rightarrow B_x \cap Z$, dilation et erosion are written

$$\delta_B(X) = (X \oplus B) \cap Z$$

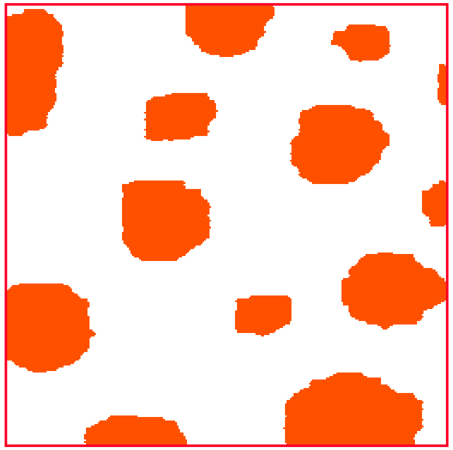
$$\varepsilon_B(X) = \{ x : B_x \cap Z \subseteq X \cap Z \}$$

The duality for the complement is formalised ***in Z***

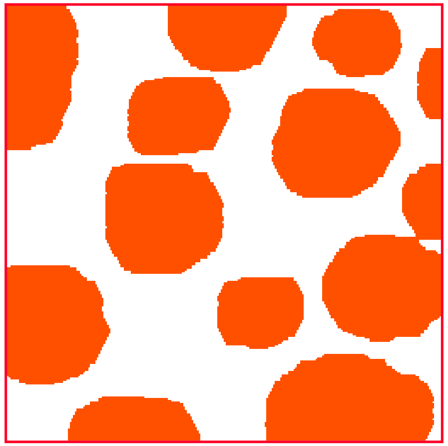
$$\psi^*(X) = Z \setminus \psi(Z \setminus X)$$

Which gives for the erosion algorithm

$$\varepsilon_B(X) = Z \setminus [\check{\delta}_B(Z \setminus X)] = [(X \cup Z^c) \ominus B] \cap Z$$



***Initial set* (X∩Z)**



***Standard Dilation of* (X∩Z)**

Kernels of the τ-mappings

- When ψ is a τ-mapping, its **kernel** $\mathcal{V}$ is defined as the set of the $Y \subseteq E$ whose transform contains the origin

$$\mathcal{V} = \{ Y,\ Y \subseteq E :\ \{o\} \in \psi(Y) \} .$$

If $\{\psi_i\}$ stands for a family of τ-applications, of kernels $\mathcal{V}_i$, the sup and the inf of the ψ_i admit $\cup\, \mathcal{V}_i$ and $\cap\, \mathcal{V}_i$ for respective kernels .

- In case of the Minkowski subtraction by B, the equality $B \ominus B = \{o\}$ implies that the corresponding kernel $\mathcal{W}_B$ be

$$\mathcal{W}_B = \{ Y,\ Y \supseteq B \} \qquad (1) .$$

On the other hand **{** ψ *increasing* } $\Leftrightarrow$ $\{\ B \in \mathcal{V},\ A \supseteq B \Rightarrow A \in \mathcal{V}\ \}$ (2).

- ***Theorem :*** *(G.Matheron, 1975)* Every increasing τ-mapping ψ on $\mathcal{P}(E)$, of kernel $\mathcal{V}$, is the following union of Minkowski subtractions

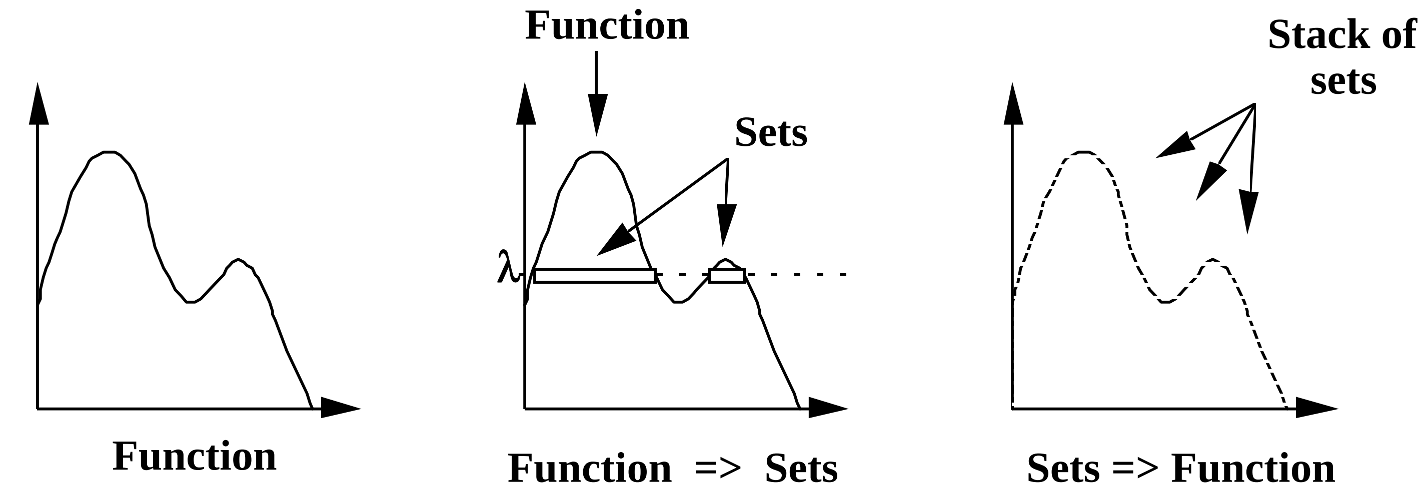
$$\psi(X) = \cup \{ X \ominus B ,\ B \in \mathcal{V} \}$$

[*derives from Eq.(1) and(2); admits a dual version for the adjoint dilation*].

Equivalence between Sets and Functions

A function can be viewed as a **stack of decreasing sets**. Each set is the intersection between the **umbra** of the function and a horizontal plane.

$$X_\lambda (f) = \{ x \in E , f(x) \geq \lambda \} \quad \Leftrightarrow \quad f(x) = \sup \{\lambda : x \in X_\lambda (f) \} \qquad (*)$$



It is equivalent to say that f is upper semi-continuous or that the X_λ's are closed. Conversely, given a family $\{X_\lambda\}$ of closed sets such that

$$\lambda \geq \mu \Rightarrow X_\lambda \subseteq X_\mu \quad \text{and} \quad X_\lambda = \cap \{X_\mu , \mu < \lambda \}$$

there exists a ***unique*** u.s.c. function f whose sections are the X_λ's.

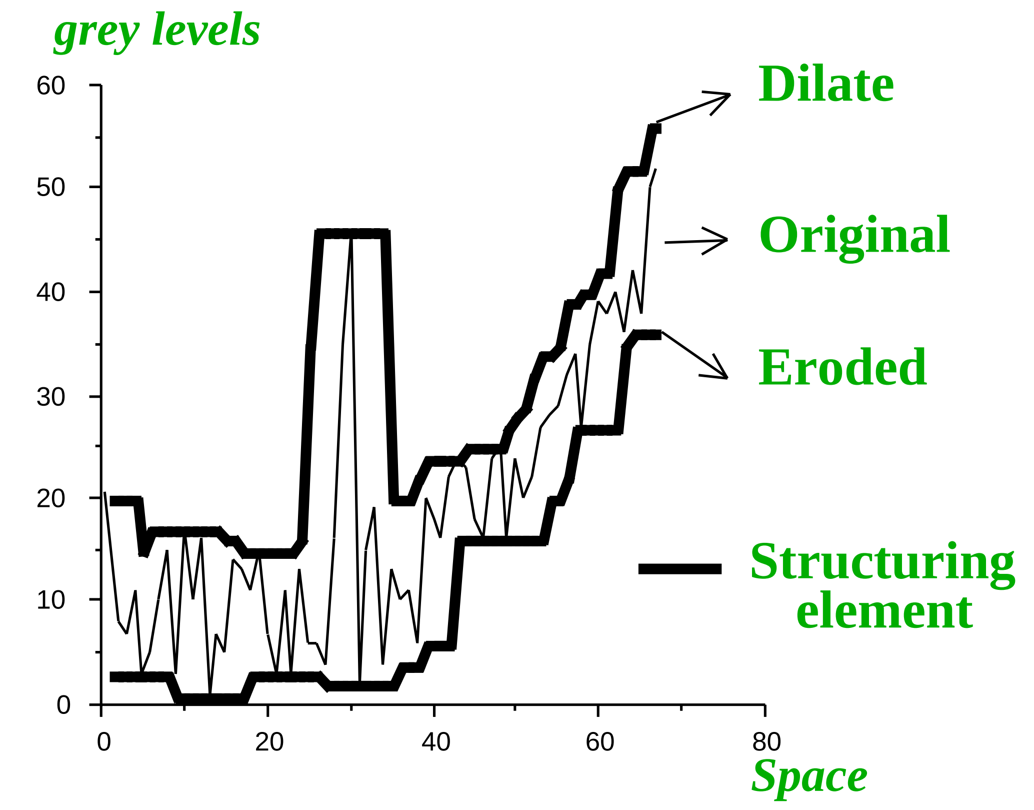
Dilation and Erosion by a flat structuring Element

Definition **:** The dilation (erosion) of a function by a flat structuring element B is introduced as the dilation (erosion) of each set $X_f(\lambda)$ by B.They are said to be **planar** .

This definition leads to the following formulae :

$$(f \oplus B)(x) = \sup\{ f(x-y),\ y \in B \}$$

$$(f \ominus B)(x) = \inf\{ f(x-y),\ -y \in B \}$$



- Erosion shrinks positive peaks. Peaks thinner that the structuring element disappear. As well, it expands the valleys and the sinks.
- Dilation produces the dual effects.

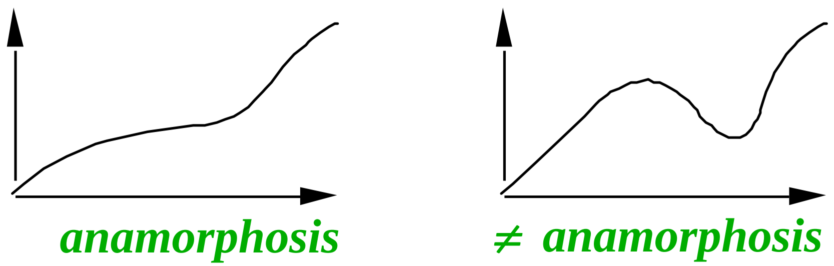
Properties of the planar operators

- Erosion and dilation, with flat or non flat structuring elements, have basically the same properties as those stated for sets.
- In addition,the use of ***flat*** structuring elements provides the three following specific advantages

Commute under anamorphosis

An anamorphosis is an increasing continuous mapping of the grey level values.

e.g. **Log (f⊕B) = (Log f)⊕B**



Stability

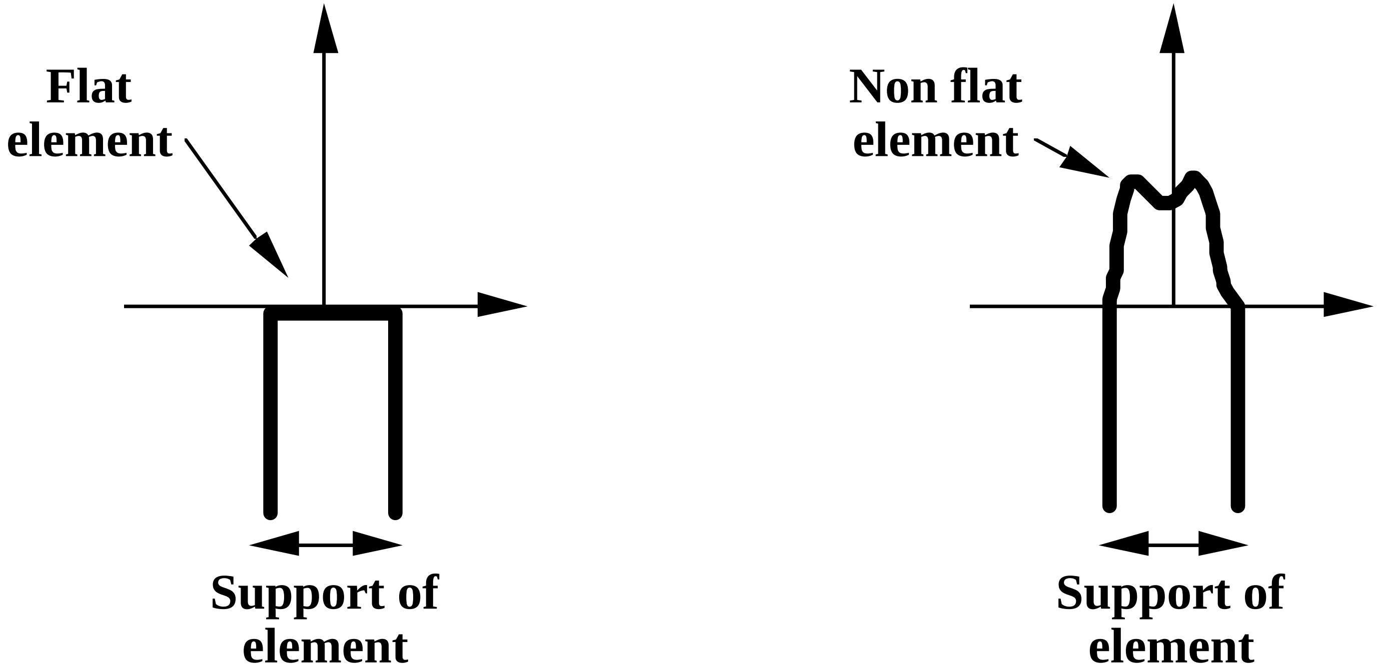
The class of the functions which take *n* given values is preserved (any n-bit image is tranformed into an n-bit image).

Implementation

A transformation based on flat structuring elements can be implemented either level by level, or numerically.

Non Planar Structuring Elements

- Planar structuring elements can be viewed as a function of constant level, equals to 0, and whose support is the structuring set. These structuring elements can be generalised by introducing weights. The resulting elements, no longer planar, are also called « ***non flat*** ».



Dilation of Functions by non flat Elements

Definition

Dilation and erosion of function f by the (non flat) function h are given by the relations

$$(f \oplus h)(x) = \sup_{y \in H} [f(x-y) + h(y)]$$

$$(f \ominus h)(x) = \inf_{-y \in H} [f(x-y) - h(y)]$$

Remark:
Since the images under study traduce physical phenomena, one shall take care to provide f and f with consistent units.

Comparison with Convolution

We can establish a parallelism between the formulae of dilation and of erosion and that of convolution .

Sum	<=>	Sup or Inf
Product	<=>	Sum

convolution :

$$h(x) * f(x) = \sum_{y \in H} f(x-y) \,.\, h(y)$$

dilation:

$$(f \oplus h)(x) = \sup_{y \in H} [f(x-y) + h(y)]$$

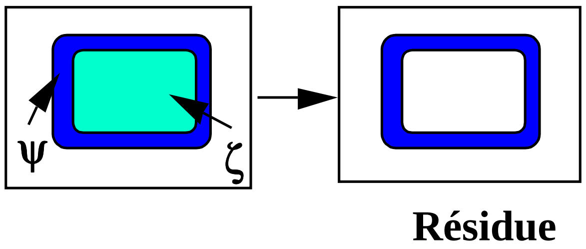
Residues of Transformations

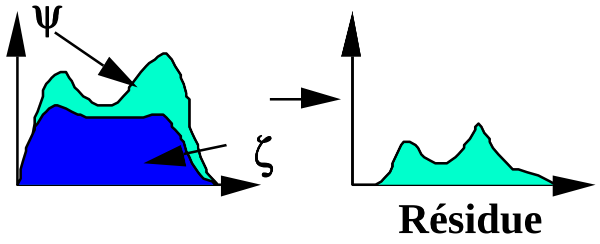
Definition

- Le residue between two transformations ψ et ζ is their difference

Set case : $\rho_{\psi,\zeta}(X) = \psi(X) \setminus \zeta(X)$

Functions case : $\rho_{\psi,\zeta}(X) = \psi(X) - \zeta(X)$





N. B. : Operations ψ and ζ may not be ordered .

Residues for Sets and Functions

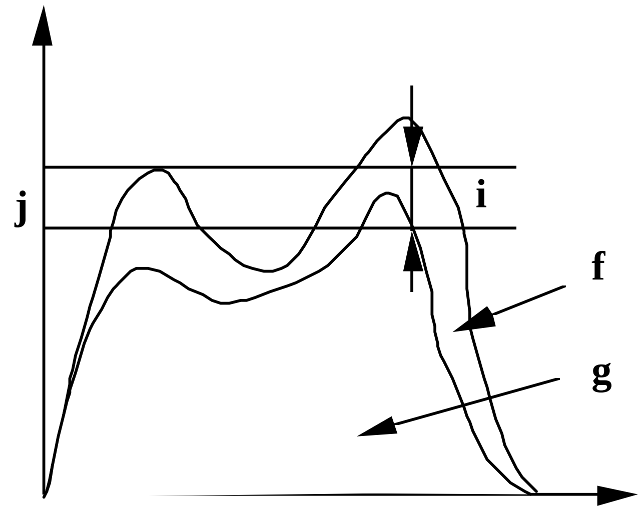
Comment

- The residues of an increasing transformation are not increasing. Therefore, in case of numerical functions, there is no level by level correspondence able to generate them, as it can be done for flat dilations or erosions.

- In the digital approach, if we put

$$X_i\ (f) = \{\ x \in\ E\ ,\ f(x) \geq i\ \}$$

then the correspondence between residues of functions and theirs stacks of section is the following (digital approach)



$$X_i\ (\ f\ -\ g\ ) = \ \cup[X_{i\ +\ k}\ (f) \setminus X_{k\ +\ 1}\ (g)\ \ ,\ \ k \geq 0\ ]$$

Morphological Gradients

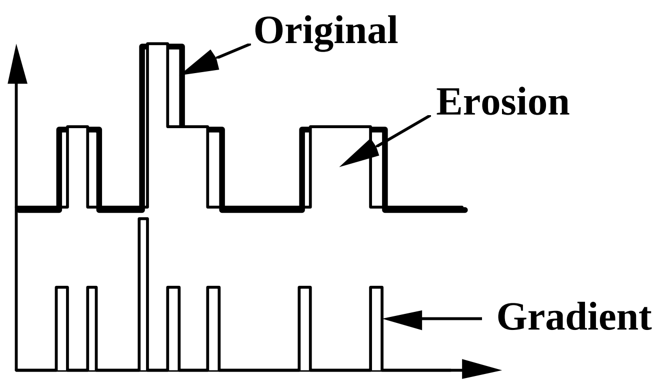
The goal of gradients transformations is to highlight contours. In digital morphology, three **Beucher's gradients** based on **the unit disc** are defined:

Gradient by erosion :

- It is the residue between the **identity** and an **erosion** , *i.e.*:

for sets $g^{-}(X) = X / (X \ominus B)$

for functions $g^{-}(f) = f - (f \ominus B)$

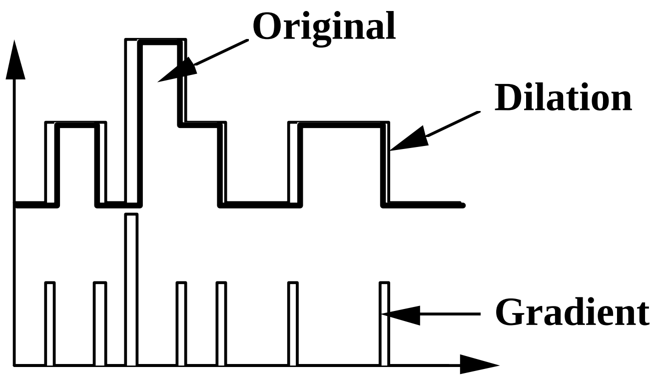


Gradient by dilation :

- It is the residue between a **dilation** and the **identity**, *i.e.* :

for sets $g^{+}(X) = (X \oplus B) / X$

for functions $g^{+}(f) = (f \oplus B) - f$



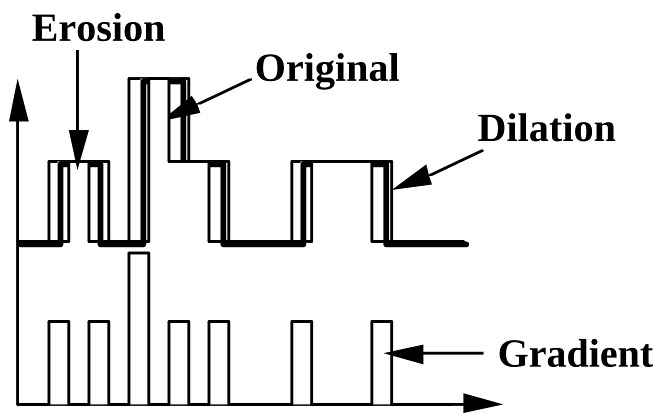
Morphological Gradients (II) and Laplacian

Symmetrical gradient :

- It is the residue between a **dilation** and an **erosion**

for sets $g(X) = (X \oplus B) / (X \ominus B)$

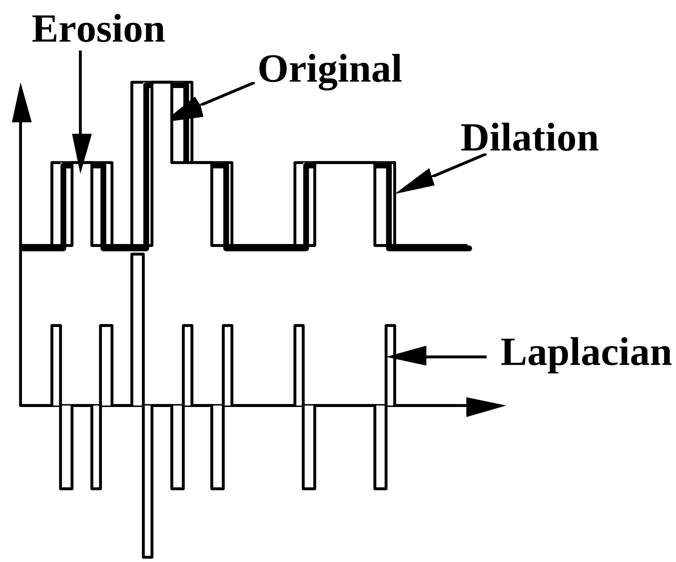
for functions $g(f) = (f \oplus B) - (f \ominus B)$



Laplacian :

- It is the **residue** between the **gradients** by dilation and erosion, for functions

$$L(f) = g^{+}(f) - g^{-}(f)$$



Note: These notions correspond the "classical" notions of gradient and laplacian (if they exist), in the limit, when the radius of disc tends towards zero.

References

On Set Dilation :

- H.Minkowski {MIN03}, in 1901, defined and studied set dilation as it is presented in chapter II. However, he did not introduce the concept of an erosion, which was defined by Hadwiger {HAD57}, in 1957.The study of the specific properties of binary dilations as a function of the geometry of the structuring element dates from the early 70's {HAA67},{SER69},{SER72}.

On Numerical Dilation :

- The extension to numerical functions began with {SER75} {ROS76}, {MEY77}, and was completed by {SER82,ch12} (semi-continuous case, flat structuring elements) and {STE86} (filters).The properties of planar mappings have been studied in length in {SER82}, {HEI91},{SOI92b}.

On Adjunction :

- Duality between erosion and dilation appears for the first time in E.Gallois's work (see {BIR84}). It was rediscovered by J.Serra in {SER88,ch.1}, who found also the representation theorems, and enriched of various properties by H.Heijmans and Ch.Ronse in {HEI90}.